Parametric Simplex Algorithms for Solving a Special Class of Nonconvex Minimization Problems

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Abstract. It is shown that parametric linear programming algorithms work efficiently for a class of nonconvex quadratic programming problems called generalized linear multiplicative programming problems, whose objective function is the sum of a linear function and a product of two linear functions. Also, it is shown that the global minimum of the sum of the two linear fractional functions over a polytope can be obtained by a similar algorithm. Our numerical experiments reveal that these problems can be solved in much the same computational time as that of solving associated linear programs. Furthermore, we will show that the same approach can be extended to a more general class of nonconvex quadratic programming problems.

Keywords. Parametric simplex algorithm, minimization of products, linear multiplicative programming.

1. Introduction

Konno and Kuno, in a recent series of articles [9], [10], showed that certain classes of nonconvex minimization problems can be solved by parametric convex minimization algorithms.

The first class of problems to which they addressed is linear multiplicative programming problems defined below:

$$\left|\begin{array}{ll} \text{minimize} & (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \\ \text{subject to} & x \in X \end{array}\right.$$

where $c_1, c_2 \in R^n$, $\sigma_1, \sigma_2 \in R^1$ and $X \subset R^n$ is a polytope. The objective function of this problem is neither (quasi-)convex nor (quasi-)concave on X. Konno and Kuno [9] divided the feasible region into two subregions

$$X_1 = X \cap \{x \in R^n \mid (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \geq 0\}$$
$$X_2 = X \cap \{x \in R^n \mid (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \leq 0\}$$

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and showed that the associated subproblems:

$$\left|\begin{array}{ll}\text{minimize} & (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \\ \text{subject to} & x \in X_1\end{array}\right. \tag{1.1}$$

$$\left|\begin{array}{ll}\text{minimize} & (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \\ \text{subject to} & x \in X_2\end{array}\right.$$

can be solved by convex minimization algorithm.

The key idea of their paper [10] is to establish the equivalence of (1.1) to its master problem:

$$\left|\begin{array}{ll}\text{minimize} & \xi(c_1^t x + \sigma)_1 + \dfrac{1}{\xi}(c_2^t x + \sigma_2) \\ \text{subject to} & x \in X_1, \quad \xi \geqslant 0\,.\end{array}\right. \tag{1.2}$$

and to apply a parametric linear programming algorithm to (1.2). Also they demonstrated that this algorithm can solve large scale linear multiplictive programming problems.

In [9], they extended this idea and proposed a similar algorithm for solving a generalized linear multiplicative programming problem:

$$\left|\begin{array}{ll}\text{minimize} & f(x) + (c_1^t x + \sigma_1)(c_2^t x + \sigma_2) \\ \text{subject to} & x \in X\end{array}\right. \tag{1.3}$$

and a generalized linear fractional programming problem:

$$\left|\begin{array}{ll}\text{minimize} & f(x) + \dfrac{(c_2^t x + \sigma_2)}{(c_1^t x + \sigma_1)} \\ \text{subject to} & x \in X\,.\end{array}\right. \tag{1.4}$$

where X is defined as before and $f(\cdot)$ is a convex function. They assumed without loss of generality that

$$(c_1^t x + \sigma_1) \geqslant 0\,, \quad (c_2^t x + \sigma_2) \geqslant 0\,, \quad \forall x \in X$$

and applied a parametric convex programming algorithm to their master problems:

$$\left|\begin{array}{ll}\text{minimize} & f(x) + \xi(c_1^t x + \sigma_1)^2 + \dfrac{1}{\xi}(c_2^t x + \sigma_2)^2 \\ \text{subject to} & x \in X\,, \quad \xi \geqslant 0\,.\end{array}\right.$$

and

$$\left|\begin{array}{ll}\text{minimize} & f(x) + \xi(c_1^t x + \sigma_1)^2 + \dfrac{1}{\xi} \cdot \dfrac{1}{(c_2^t x + \sigma_2)} \\ \text{subject to} & x \in X\,, \quad \xi \geqslant 0\end{array}\right.$$

Also they applied this algorithm to a special case of (1.3) in which $f(\cdot)$ is a quadratic function and claimed that it is reasonably efficient.

The purpose of this paper is to propose an alternative algorithm for solving (1.3) and (1.4) when $f(\cdot)$ is affine or quadratic. It will be shown in Section 2 that the former problem can be reformulated as a linear programming problem containing a parameter in both its objective function and its right hand side vector. This reformulation enables us to construct a variant of parametric simplex algorithm. Section 3 will be devoted to the extension of the algorithm to the case where $f(\cdot)$ is a convex quadratic function. It appears that our approach is simpler and at least as efficient as the ones proposed in [14]. In Section 4, we will show that the algorithm developed in Section 2 can be adapted to a generalized linear fractional programming problem (1.4) in which $f(\cdot)$ is a linear fractional function.

It should be emphasized that these nonconvex minimization problems have important applications in economics [6], bond portfolio optimization [8], and so forth. Readers are referred to [12, 13] for advanced development in nonconvex quadratic programming problems.

Results of numerical experiments of our algorithm will be presented in Section 5. Finally in the Appendix, we will discuss the way to get around degeneracy.

2. Parametric Simplex Algorithm for Minimizing the Sum of a Linear Function and a Linear Multiplicative Function

2.1. PARAMETRIC MASTER PROBLEM

Let us consider a special type of quadratic programming problem defined below:

$$\left|\begin{array}{ll} \text{minimize} & G(x) = d^t x + c^t x \cdot g^t x , \\ \text{subject to} & Ax = b , \quad x \geq 0 . \end{array}\right. \tag{2.1}$$

where c, d, $g \in R^n$, $A \in R^{m \times n}$, $b \in R^m$. Let us assume for simplicity that the feasible region

$$X = \{x \in R^n \mid Ax = b, \quad x \geq 0\}$$

is non-empty and bounded.

$G(\cdot)$ is neither (quasi-)convex nor (quasi-)concave on X, so that it can have multiple local minima as demonstrated in [10], [14]. To solve (2.1), we first introduce an auxiliary variable

$$\xi = g^t x ,$$

and define a master problem:

$$\left|\begin{array}{ll} \text{minimize} & F(x; \xi) = d^t x + \xi \cdot c^t x \\ \text{subject} & Ax = b , \quad x \geq 0 , \\ & g^t x = \xi , \quad \xi_{\min} \leq \xi \leq \xi_{\max} . \end{array}\right. \tag{2.2}$$

where

$$\begin{cases} \xi_{\min} = \min\{g^t x \mid Ax = b, \quad x \geq 0\} \\ \xi_{\max} = \max\{g^t x \mid Ax = b, \quad x \geq 0\} \end{cases}$$

Problem (2.2) has an optimal solution since its feasible region is nonempty and bounded.

THEOREM 2.1. *Let* (x^*, ξ^*) *be an optimal solution of (2.2). Then* x^* *is an optimal solution of (2.1).*

Proof. Obvious from the definition of (2.2). □

This reformulation leads us to apply parametric linear programming approach to solve (2.1). Let us introduce a class of linear programs:

$$P(\xi) \left| \begin{array}{ll} \text{minimize} & (d + \xi c)^t x \\ \text{subject to} & \tilde{A}x = \begin{pmatrix} b \\ \xi \end{pmatrix}, \quad x \geq 0 . \end{array} \right.$$

where $\xi \in [\xi_{\min}, \xi_{\max}]$ and

$$\tilde{A} = \begin{pmatrix} A \\ g^t \end{pmatrix} = \begin{pmatrix} a_1, a_2, \ldots, a_n \\ g_1, g_2, \ldots, g_n \end{pmatrix} .$$

Let $x(\xi)$ be an optimal solution of $P(\xi)$ and let

$$h(\xi) = (d + \xi c)^t x(\xi) .$$

Also let

$$\xi^* = \operatorname{argmin}\{h(\xi) \mid \xi_{\min} \leq \xi \leq \xi_{\max}\} .$$

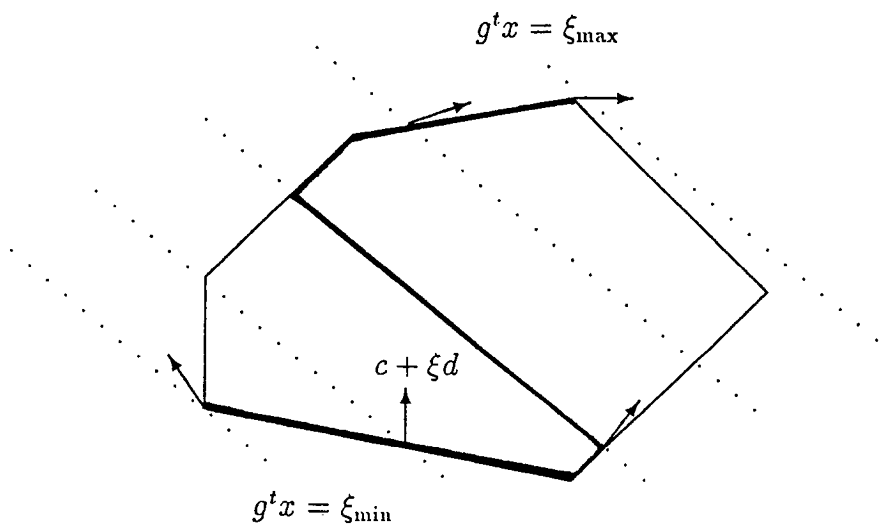

Fig. 1. The trajectory of $x^*(\xi)$.

Then $x^* \equiv x(\xi^*)$ is an optimal solution of (2.1). Figure 1 shows a possible trajectory of $x^*(\xi)$.

2.2. ALGORITHM FOR SOLVING THE MASTER PROBLEM

Let $x(\xi_0)$ be an optimal basic solution of $P(\xi_0)$, where $\xi_0 \in [\xi_{\min}, \xi_{\max}]$ and let B be an associated basis matrix of $\tilde{A}$ (we assume for simplicity that matrix $\tilde{A}$ has full row rank). Also let

$$\begin{cases} \tilde{A} = (B, N)\,, \\ x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} d = \begin{pmatrix} d_B \\ d_N \end{pmatrix} \end{cases}$$

be the partition of a matrix and vectors corresponding to the basis matrix B.

By using the familiar notations

$$\begin{cases} \pi = c_B^t B^{-1}\,, \quad \sigma = d_B^t B^{-1}\,, \\ \bar{c}_N^t = c_N^t - \pi N\,, \quad \bar{d}_N^t = d_N^t - \sigma N\,, \\ \bar{b}(\xi_0) = B^{-1}\begin{pmatrix} b \\ \xi_0 \end{pmatrix}, \\ \bar{N} = B^{-1}N\,, \end{cases}$$

we obtain the optimal dictionary of $P(\xi_0)$:

$$\begin{array}{|ll} \text{minimize} & (d_B + \xi_0 c_B)^t \bar{b}(\xi_0) + (\bar{d}_N + \xi_0 \bar{c}_N)^t x_N \\ \text{subject to} & x_B = \bar{b}(\xi_0) - \bar{N} x_N\,, \\ & x_B \geq 0\,, \quad x_N \geq 0\,. \end{array}$$

Note that $\bar{d}_N + \xi_0 \bar{c}_N \geq 0$ and $\bar{b}(\xi_0) \geq 0$.

THEOREM 2.2. *B is an optimal basis of $P(\xi)$ for all ξ satisfying the following conditions*

$$\bar{d}_N + \xi \bar{c}_N \geq 0\,, \tag{2.3}$$

$$B^{-1}\begin{pmatrix} b \\ \xi \end{pmatrix} \geq 0\,. \tag{2.4}$$

Proof. See [3], [5]. □

The condition (2.3) generates an interval $[\alpha_1, \beta_1]$ where

$$\begin{cases} \alpha_1 = \max\{-\bar{d}_j/\bar{c}_j \mid \bar{c}_j > 0\} \\ \beta_1 = \min\{-\bar{d}_j/\bar{c}_j \mid \bar{c}_j < 0\} \end{cases} \tag{2.5}$$

where $\bar{c}_j$, $\bar{d}_j$'s are components of $\bar{c}_N$ and $\bar{d}_N$. Also, by writing

$$B^{-1}\begin{pmatrix} b \\ \xi \end{pmatrix} = q + \xi p\,, \tag{2.6}$$

the condition (2.4) is equivalent to $\alpha_2 \leq \xi \leq \beta_2$ where

$$\begin{cases} \alpha_2 = \max\limits_{1\leq i\leq m+1} \{-q_i/p_i \mid p_i > 0\}\,, \\ \beta_2 = \min\limits_{1\leq i\leq m+1} \{-q_i/p_i \mid p_i < 0\}\,. \end{cases} \tag{2.7}$$

Hence B is an optimal basis of $P(\xi)$ for all $\xi \in [\underline{\xi}, \bar{\xi}]$ where

$$\begin{cases} \underline{\xi} = \max\{\alpha_1, \alpha_2\}\,, \\ \bar{\xi} = \min\{\beta_1, \beta_2\}\,. \end{cases} \tag{2.8}$$

Note that $[\underline{\xi}, \bar{\xi}]$ is nonempty since $\xi_0 \in [\underline{\xi}, \bar{\xi}]$.

Let us impose here the non-degeneracy assumption (see Appendix for the detailed discussion about degeneracy).

NON-DEGENERACY ASSUMPTION. *The following conditions hold for all* $\xi_0 \in [\xi_{\min}, \xi_{\max}]$

(*i*) $\underline{\xi} < \bar{\xi}$,
(*ii*) $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$.

Case 1. $\bar{\xi} = \beta_1$.
When ξ reaches $\bar{\xi}$, we obtain an alternative optimal dictionary of $P(\bar{\xi})$ by choosing the nonbasic column vector $(a_r^t, g_r)^t$ of $\tilde{A}$ as an incoming basic vector where

$$\beta_1 = -\bar{d}_r/\bar{c}_r\,.$$

Alternative optimal dictionary of $P(\bar{\xi})$ can be obtained by a primal simplex pivoting procedure. Note that we can find a pivot element since the feasible region of $P(\bar{\xi})$ is bounded. Let B' be the alternative optimal basis.

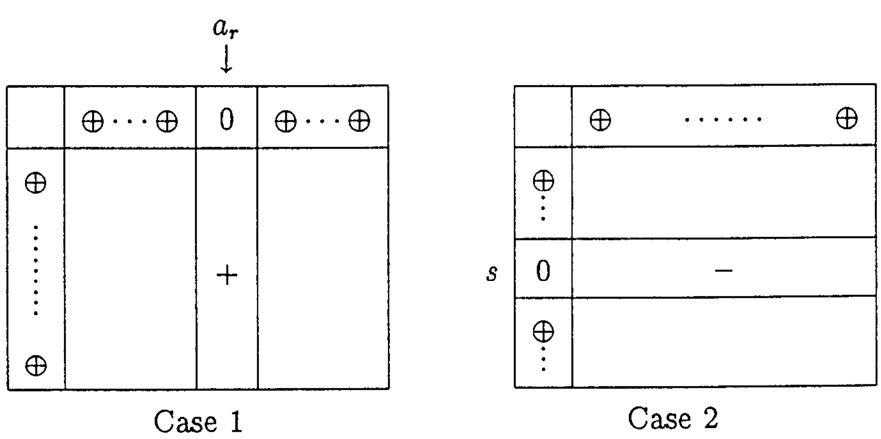

Fig. 2. Optimal dictionary for $P(\bar{\xi})$.

Case 2. $\bar{\xi} = \beta_2$.
At $\xi = \bar{\xi}$, we obtain an alternative optimal dictionary of $P(\bar{\xi})$ by performing a single dual simplex pivoting at the row s where

$$\beta_2 = -q_s/p_s \, .$$

Let B' be the alternative optimal basis. Let $[\bar{\xi}, \bar{\xi}']$ be the interval of ξ in which B' is optimal.

At this juncture, we will prove the condition which guarantees the termination of our procedure.

THEOREM 2.3. *In case $\bar{\xi} = \beta_2$ and all the element of the s-th row of the optimal dictionary of $P(\bar{\xi})$ are nonpositive, then $\bar{\xi} = \xi_{\max}$.*

Proof. Let us write the s-th row of the dictionary as follows.

$$x_{B_s} = \bar{b}_s(\xi) - \sum_{j \in J_N} \bar{a}_{sj} x_j \, .$$

By assumption $b_s(\xi) < 0$ for $\xi > \bar{\xi}$, so that $P(\xi)$ is infeasible for $\xi > \bar{\xi}$. By definition $P(\xi)$ is feasible for all $\xi \in [\xi_{\min}, \xi_{\max}]$, which proves $\bar{\xi} = \xi_{\max}$. □

Starting from the optimal basis for $P(\xi_{\min})$, we can generate a sequence of constants

$$\xi_{\min} \equiv \xi_1 < \xi_2 < \cdots < \xi_{k+1} \equiv \xi_{\max} \, ,$$

and a sequence of bases $B_1, B_2, \ldots, B_k$ such that B_j is optimal for all $P(\xi)$, $\xi \in [\xi_j, \xi_{j+1}]$.

Let

$$h_j = \min\{(c_{B_j} + \xi d_{B_j})^t \bar{b}(\xi) \mid \xi \in [\xi_j, \xi_{j+1}]\} = (c_{B_j} + \xi^j d_{B_j})^t \bar{b}(\xi^j) \, ,$$

and let

$$h_l = \min_{1 \le j \le k} h_j \, ,$$

Then

$$\begin{cases} x^*_{B_l} = \bar{b}(\xi^l) \\ x^*_{N_l} = 0 \end{cases}$$

is a global minimum of (2.1). Figure 4 shows the behavior of $h(\xi)$, and the dotted line shows the value of h_j associated with the interval $[\xi_j, \xi_{j+1}]$.

Algorithm

Stage 1. Solve a linear program

$$\left| \begin{array}{ll} \text{minimize} & g^t x \\ \text{subject to} & x \in X \end{array} \right.$$

and let $\xi_{\min} = \min\{g'x \mid x \in X\}$. Also let $\hat{x}$ be an optimal solution of $P(\xi_{\min})$ and let B_1 be an optimal basis of $P(\xi_{\min})$. $t = 1$.

Stage 2. Let $x^* = \hat{x}$, $h^* = \infty$, $\underline{\xi} = \xi_{\min}$.

(1) Calculate $\bar{\xi}$ according to (2.5), (2.7) and (2.8).

(2) Let

$$\begin{aligned} v &= (c_{B_t} + \hat{\xi} d_{B_t})'\bar{b}(\hat{\xi}) \\ &= \min\{(c_{B_t} + \xi d_{B_t})'\bar{b}(\xi) \mid \xi \in [\underline{\xi}, \bar{\xi}]\} \end{aligned}$$

If $v < v^*$, then let $v^* = v$, $x^*_{B_t} = \bar{b}(\hat{\xi})$, $x^*_{N_t} = 0$, where N_t stands for the matrix of nonbasic columns associated with B_t.

(3) *case 1.* $\bar{\xi} = \beta_1$. Obtain a new optimal dictionary associated with an alternative optimal basis B_{t+1} of $P(\bar{\xi})$ by applying a primal simplex procedure at column r where $\beta_1 = -\bar{d}_r/\bar{c}_r$. Let $t = t + 1$ and go to (1).

case 2. $\bar{\xi} = \beta_2$. Obtain a new optimal dictionary associated with an alternative optimal basis B_{t+1} of $P(\bar{\xi})$ by applying, if possible, a dual simplex procedure at row s where $\beta_2 = -q_s/p_s$. Let $t = t + 1$ and go to (3). If such a basis cannot be found, then terminate. □

2.3. LINEAR MULTIPLICATIVE PROGRAMMING PROBLEMS

Let us consider the special case of (2.1) in which $d = 0$. This is nothing but a linear multiplicative programming problem treated in [1], [2], [10], [14]. In this case, problem (2.2) reduces to

$$\left| \begin{array}{ll} \text{minimize} & \xi \cdot c'x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,, \\ & g'x = \xi\,, \quad \xi_{\min} \leq \xi \leq \xi_{\max}\,. \end{array} \right.$$

Thus it sufficies to solve two parametric right hand side linear programs

$$\left| \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,, \\ & g'x = \xi\,, \quad 0 \leq \xi \leq \xi_{\max}\,, \end{array} \right.$$

and

$$\left| \begin{array}{ll} \text{maximize} & c'x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,, \\ & g'x = \xi\,, \quad \xi_{\min} \leq \xi \leq 0\,. \end{array} \right.$$

Solving these problems is easier than solving (2.2). Also it looks simpler than the algorithm proposed in [10].

3. Parametric Linear Complementary Problems Associated with Generalized Linear Multiplicative Programming Problems

The algorithm presented in Section 2 can be extended to a more general class of nonconvex quadratic programming problems:

$$\left|\begin{array}{ll}\text{minimize} & G(x) = d^t x + \frac{1}{2} x^t D x + c^t x \cdot g^t x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,.\end{array}\right. \tag{3.1}$$

where D is a symmetric positive semi-definite matrix and the feasible region

$$X = \{x \in R^n \mid Ax = b\,, \quad x \geq 0\}$$

is nonempty and bounded. It is now straightforward to see that (3.1) is equivalent to the following master problem:

$$\left|\begin{array}{ll}\text{minimize} & F(x; \xi) = d^t x + \frac{1}{2} x^t D x + \xi \cdot c^t x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,, \\ & g^t x = \xi\,, \quad \xi_{\min} \leq \xi \leq \xi_{\max}\,.\end{array}\right.$$

where

$$\xi_{\min} = \min\{g^t x \mid Ax = b\,, \quad x \geq 0\}$$
$$\xi_{\max} = \max\{g^t x \mid Ax = b\,, \quad x \geq 0\}$$

Let us choose $\xi_0 \in [\xi_{\min}, \xi_{\max}]$ and consider a quadratic programming problem:

$$Q(\xi_0)\left|\begin{array}{ll}\text{minimize} & F(x; \xi_0) = d^t x + \frac{1}{2} x^t D x + \xi_0 c^t x \\ \text{subject to} & Ax = b\,, \quad x \geq 0\,, \\ & g^t x = \xi_0\,.\end{array}\right.$$

$Q(\xi_0)$ is a convex quadratic programming problem and hence is equivalent to the following linear complementarity problem [4], [11].

$$LCP(\xi_0)\left|\begin{array}{l}\text{Find vectors} \\ \quad x \in R^m_+\,, \quad u \in R^n_+\,, \quad y \in R^m_+\,, \\ \quad v \in R^m_+\,, \quad z_1 \in R^1_+\,, \quad z_2 \in R^1_+\,, \qquad (3.2) \\ \text{such that} \\ \begin{pmatrix} D & -A^t & -g & g & -I & 0 \\ A & 0 & 0 & 0 & 0 & -I \\ g^t & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z_1 \\ z_2 \\ u \\ v \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}\xi_0 + \begin{pmatrix} -d \\ b \\ 0 \end{pmatrix}, \\ u^t x = 0\,, \quad v^t y = 0\,. \qquad (3.3)\end{array}\right.$$

By using a standard method [11], we can obtain an optimal basic solution $w^* \equiv (x^*, y^*, z_1^*, z_2^*, u^*, v^*)$ of the systems of equations (3.2) such that $(u^*)^t x^* = 0$ and $(v^*)^t y^* = 0$. Let B be an optimal complementary basis of (3.2). Also let w_B, w_N be the associated basic and nonbasic variables, respectively. Then the complementary dictionary can be written as follows:

$$w_B = \bar{q} + \bar{p}\xi_0 - \bar{N}w_N ,$$

where $\bar{q} + \bar{p}\xi_0 \geqslant 0$. This dictionary is optimal for all ξ such that

$$\bar{q} + \bar{p}\xi \geqslant 0 ,$$

from which we obain an interval $[\underline{\xi}, \bar{\xi}]$. When ξ reaches $\bar{\xi}$, one of the basic variables, say w_s becomes zero. We will replace w_s with a nonbasic variable w_t according to the following rule:

(i) If $w_s = x_l$ then $w_t = u_l$,
(ii) If $w_s = u_l$ then $w_t = x_l$,
(iii) If $w_s = y_r$ then $w_t = v_r$,
(iv) If $w_s = v_r$ then $w_t = y_r$,
(v) If $w_s = z_1$ then $w_t = z_2$,
(vi) If $w_s = z_2$ then $w_t = z_1$.

This exchange rule maintains the complementarity condition (3.3), so that we obtain a dictionary associated with a complementary basic solution of $LCP(\bar{\xi})$. Let B' be the new complementary basis. Associated with B', we calculate the interval $[\underline{\xi}', \bar{\xi}']$ in which B' is optimal (note that $\underline{\xi}' = \bar{\xi}$). Choosing $\xi_0 = \xi_{\min}$ to start with, we will obtain a sequence of constants

$$\xi_{\min} < \xi_1 < \xi_2 < \ldots, < \xi_{k+1} = \xi_{\max} ,$$

and a sequence of complementary bases $B_1, B_2, \ldots, B_k$ by avoiding cycling due to degeneracy. Let

$$F_j = \min\{F(x^*(\xi); \xi) \mid \xi_j \leqslant \xi \leqslant \xi_{j+1}\} .$$

Note that $x^*(\xi)$ is a linear function of ξ in the interval $[\xi_j, \xi_{j+1}]$, so that $F(x^*(\xi), \xi)$ is a quadratic function of ξ. Thus F_j can be calculated by elementary arithmetic. Obviously

$$F_l \equiv \min_{1 \leqslant j \leqslant k} F_j$$

gives the global minimum of $f(x)$ over X. Also $x^*(\xi^l)$ is an optimal solution of (3.1) where

$$\xi^l = \operatorname{argmin}\{F(x^*(\xi); \xi) \mid \xi_l \leqslant \xi \leqslant \xi_{l+1}\} .$$

4. Minimizing the Sum of Two Linear Fractional Functions

Let us consider here the following nonconvex programming problem [8]

$$\left|\begin{array}{ll}\text{minimize} & F(x)=\dfrac{c_1^t x+\sigma_1}{c_2^t x+\sigma_2}+\dfrac{d_1^t x+\tau_1}{d_2^t x+\tau_2} \\ \text{subject to} & Ax=b\,, \quad x\geqslant 0\,.\end{array}\right. \tag{4.1}$$

Figure 3 shows the three-dimensional picture of the sum of two linear fractional function with two variables, which shows that $F(\cdot)$ is neither convex nor concave. We will assume that

$$c_2^t x+\sigma_2>0\,, \quad d_2^t x+\tau_2>0\,, \quad \forall x\in X\,, \tag{4.2}$$

where

$$X=\{x\in R^n \mid Ax=b\,, \quad x\geqslant 0\}\,.$$

Let

$$y_0=\frac{1}{d_2^t x+\tau_2}\,.$$

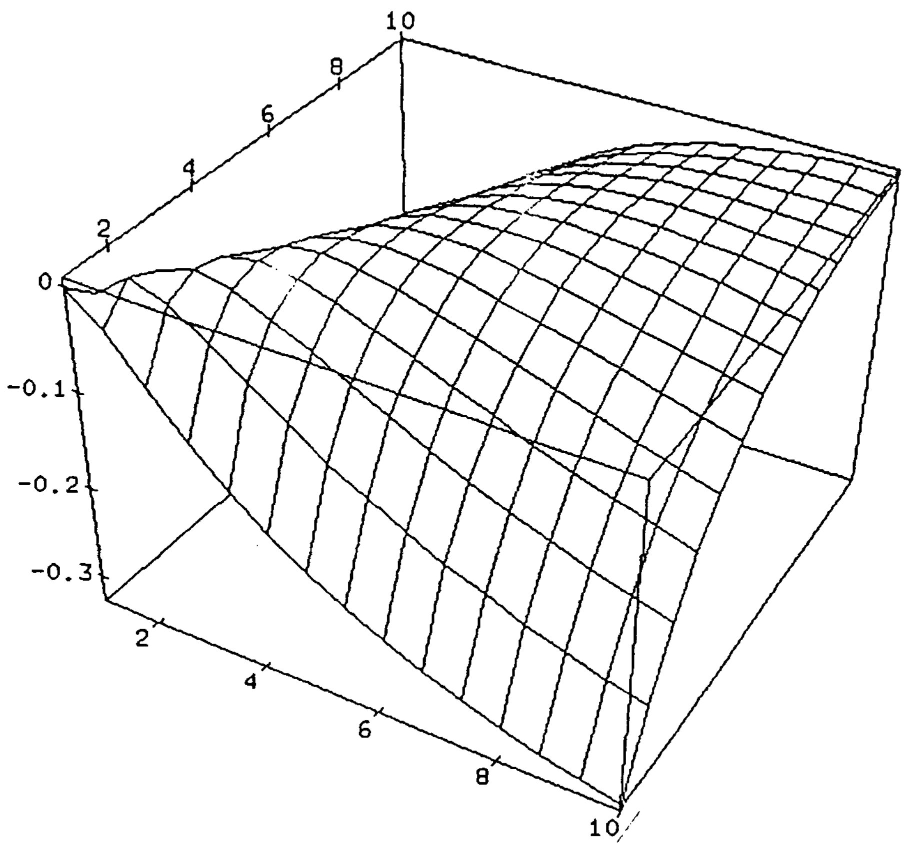

Fig. 3. 3-dimensional plot of the function $F(x)$ of (4.1).

Then the problem (4.1) is equivalent to

$$\left|\begin{array}{ll} \text{minimize} & \tilde{F}(x;\, y_0) = \dfrac{(c_1^t x + \sigma_1)y_0}{(c_2^t x + \sigma_2)y_0} + (d_1^t x + \tau_1)y_0 \\ \text{subject to} & Ax \cdot y_0 - by_0 = 0\,, \\ & y_0(d_2^t x + \tau_2) = 1\,, \\ & x \geqslant 0\,, \quad y_0 \geqslant 0\,. \end{array}\right. \tag{4.3}$$

Let

$$y = y_0 \cdot x\,.$$

Then (4.3) is equivalent to

$$\left|\begin{array}{ll} \text{minimize} & \hat{F}(y,\, y_0) = \dfrac{c_1^t y + \sigma_1 y_0}{c_2^t y + \sigma_2 y_0} + d_1^t y + \tau_1 y_0 \\ \text{subject to} & Ay - by_0 = 0 \\ & d_2^t y + \tau_2 y_0 = 1\,, \\ & y \geqslant 0\,, \quad y_0 \geqslant 0\,. \end{array}\right.$$

Let

$$\begin{aligned} \xi_{\min} &= \inf\{c_2^t y + \sigma_2 y_0 \mid Ay - by_0 = 0\,, \quad d_2^t y + \tau_2 y_0 = 1\,, \\ &\qquad y \geqslant 0\,, \quad y_0 \geqslant 0\}\,, \\ \xi_{\max} &= \sup\{c_2^t y + \sigma_2 y_0 \mid Ay - by_0 = 0\,, \quad d_2^t y + \tau_2 y_0 = 1\,, \\ &\qquad y \geqslant 0\,, \quad y_0 \geqslant 0\}\,. \end{aligned}$$

PROPOSITION 4.1. $0 < \xi_{\min} \leqslant \xi_{\max} < \infty$.

Proof. If $\xi_{\min} \leqslant 0$, then there exits $(y,\, y_0) \geqslant 0$ satisfying $Ay - by_0 = 0$ such that $c_2^t y + \sigma_2 y_0 \leqslant 0$. By definition $y_0 = 1/(d_2^t x + \tau_2)$ for some $x \in X$. Hence $y_0 > 0$, so that

$$A(y/y_0) - b = 0\,, \quad c_2^t(y/y_0) + \sigma_2 \leqslant 0\,, \quad y/y_0 \geqslant 0\,.$$

This is a contradiction to the assumption (4.2). Thus $\xi_{\min} > 0$. $\xi_{\max} < \infty$ can be proved analogously. □

Let us consider a parametric linear programming problem:

$$\left|\begin{array}{ll}\text{minimize} & \frac{1}{\xi}(c_1^t y + \sigma_1 y_0) + (d_1^t y + \tau_1 y_0) \\ \text{subject to} & Ax - by_0 = 0\,, \\ & d_2^t y + \tau_2 y_0 = 1\,, \\ & c_2^t y + \sigma_2 y_0 = \xi\,, \\ & y \geqslant 0\,, \quad y_0 \geqslant 0\,, \\ & \xi_{\min} \leqslant \xi \leqslant \xi_{\max}\,. \end{array}\right. \tag{4.4}$$

THEOREM 4.2. (*i*) *If* y^*, y_0^* *is an optimal solution of* (*4.4*), *then* $x^* = y^*/y_0^*$ *is an optimal solution of* (*4.1*).

(*ii*) *If* (*4.4*) *has an unbounded solution, then* (*4.1*) *has an unbounded solution.*

Proof. Obvious. □

Minor modification of the algorithm in Section 2 works for this problem.

5. Computational Experiments

We will report the results of the computational experiments of the algorithm presented in Section 2. The program was coded in C language and tested on a SUN4/280S computer.

We solved the problems of the form:

$$\left|\begin{array}{ll}\text{minimize} & G(x) = d^t x + c^t x \cdot g^t x \\ \text{subject to} & Ax \geqslant b\,, \quad x \geqslant 0\,. \end{array}\right.$$

where c, d, $g \in R^n$, $A \in R^{m\times n}$, $b \in R^m$. All elements of A, b, c, d and g were randomly generated, whose ranges are [0, 100]. This implies that every problem has a finite optimal solution, since its feasible region is bounded.

Ten examples were solved for each size of the problems. Table I shows the remarkable performance of our algorithm. In fact, Stage 2 requires much less CPU time than Stage 1 (about 5% of Stage 1) for all problems.

We also tested randomly generated problems of the form:

$$\left|\begin{array}{ll}\text{minimize} & G(x) = d^t x + \dfrac{c^t x}{g^t x} \\ \text{subject to} & Ax \geqslant b\,, \quad x \geqslant 0\,. \end{array}\right.$$

where c, d, $g \in R^n$, $A \in R^{m\times n}$, $b \in R^m$. The algorithm similar to the one presented in Section 2 solved the above problem efficiently. Our numerical experiments reveal that these problems can be solved in much the same computational time as that of solving associated linear programs.

Table I. CPU time (in seconds) and number of pivots

m	150	150	200	200	250	250	300	300	350
n	100	150	150	200	200	250	250	300	300
Average CPU time (standard deviation)									
Stage 1	46.2	61.5	113.2	137.5	220.8	300.9	407.0	432.9	610.9
	(8.2)	(10.7)	(17.2)	(22.4)	(34.1)	(49.9)	(55.4)	(92.5)	(113.7)
Stage 2	2.6	4.2	5.5	8.4	10.1	16.0	19.0	21.1	26.2
	(0.7)	(1.1)	(1.4)	(1.4)	(2.5)	(4.2)	(4.4)	(7.2)	(9.8)
Total	48.8	65.7	118.7	145.9	230.9	316.9	416.0	454.0	637.1
Average Number of Pivots (standard deviation)									
Stage 1	266.2	290.5	341.1	362.1	413.7	507.7	510.0	507.0	567.2
	(48.5)	(50.8)	(52.1)	(59.5)	(63.8)	(86.0)	(73.3)	(108.5)	(106.8)
Stage 2	29.9	33.0	33.2	37.1	37.6	48.0	48.1	44.8	47.9
	(8.1)	(8.8)	(8.7)	(6.3)	(9.2)	(12.4)	(11.1)	(15.2)	(17.6)
Total	296.1	323.5	374.3	399.2	451.3	555.7	558.1	551.8	615.1

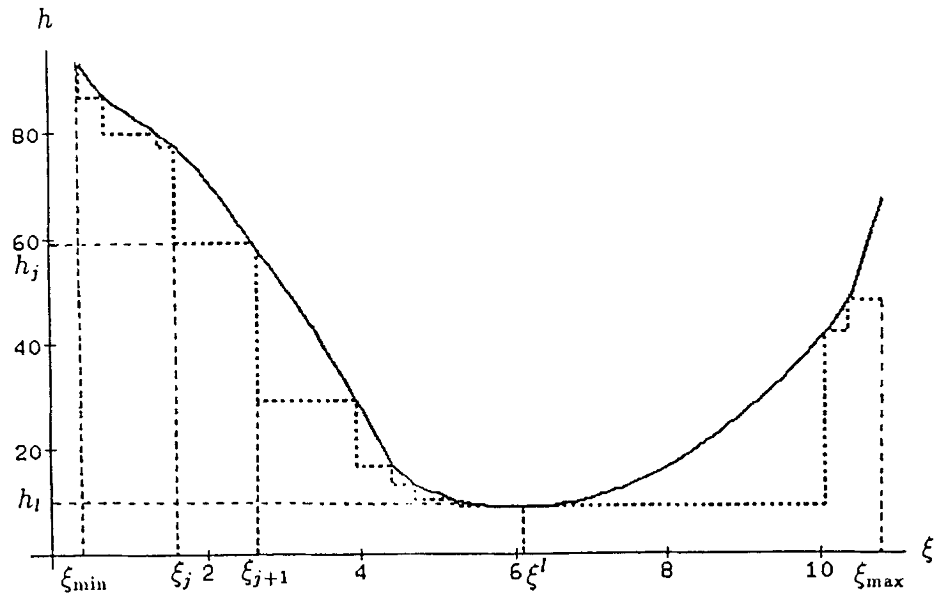

Fig. 4. Behavior of $h(\xi)$.

6. Appendix

Here we discuss pivoting rules which avoid cycling in the case of degeneracy.

First we construct a pivoting rule similar to the criss-cross method for Problem 2.1. For any pair of vectors $x = (x_1, \ldots, x_k)$ and $x' = (x'_1, \ldots, x'_k)$, we say x is *lexicographically greater than or equal to* x', denoted by $x \geqslant_{lex} x'$, or $x' \leqslant_{lex} x$, if there exists i $(1 \leqslant i \leqslant k)$ satisfying that $x_j = x'_j$, $1 \leqslant^{\forall} j < i$ and $x_i > x'_i$ or $x = x'$. Given a scalar ξ_0 with $\xi_{\min} \leqslant \xi_0 \leqslant \xi_{\max}$, our purpose is to obtain a basis satisfying

$$(\bar{d}_j + \xi_0 \bar{c}_j, \bar{c}_j) \geqslant_{lex} \mathbf{0} \quad \text{and} \quad (q_j + \xi_0 p_j, p_j) \geqslant_{lex} \mathbf{0}\,, \quad \text{for all } j\,, \tag{6.1}$$

where $\mathbf{0} = (0, 0)$. If a basis satisfying (6.1) obtained, then $\bar{\xi} > \xi_0$ and we will update ξ_0 by $\bar{\xi}$.

Given a basis B, we define the scalar $\Delta(B)$ as:

$$\Delta(B) = \min\Bigg\{\min\bigg\{\frac{|\bar{d}_j + \xi_0 \bar{c}_j|}{|\bar{c}_j|} : j \text{ is nonbasic index}\,,\ |\bar{d}_j + \xi_0 \bar{c}_j| > 0 \ \text{ and } \ |\bar{c}_j| > 0\bigg\}\,, \min\bigg\{\frac{|q_j + \xi_0 p_j|}{|p_j|} : j \text{ is basic index}\,,\ |q_j + \xi_0 p_j| > 0 \ \text{ and } \ |p_j| > 0\bigg\}\Bigg\}\,.$$

Let $\Delta_{\min} = \frac{1}{2} \min\{\Delta(B) : B \text{ is a basis of } \tilde{A}\}$. Then from the definition of $\Delta(B)$, $\Delta_{\min} > 0$. The definition of $\Delta_{\min}$ directly implies the following lemma.

LEMMA 6.1. *For any basis B of $\tilde{A}$ and for any index j,*

$$(\bar{d}_j + \xi_0 \bar{c}_j, \bar{c}_j) \geqslant_{lex} \mathbf{0} \text{ if and only if } \bar{d}_j + (\xi_0 + \Delta_{\min})\bar{c}_j \geqslant 0 ,$$
$$(q_j + \xi_0 p_j, p_j) \geqslant_{lex} \mathbf{0} \text{ if and only if } q_j + (\xi_0 + \Delta_{\min})p_j \geqslant 0 .$$

By solving a linear program $P(\xi_0 + \Delta_{\min})$ with a suitable method, we obtain an optimal, infeasible or dual infeasible basis. In case an infeasible basis is obtained, then $\xi_0 = \xi_{\max}$ as was shown in Theorem 2.3. Assume that a dual infeasible basis is obtained. It implies that $P(\xi_0 + \Delta_{\min})$ is dual infeasible. If $P(\xi_0 + \Delta_{\min})$ is not infeasible, then $P(\xi_0 + \Delta_{\min})$ is unbounded and it contradicts with the boundedness of the feasible region of $P(\xi_0 + \Delta_{\min})$. Thus $P(\xi_0 + \Delta_{\min})$ is infeasible and there exists an infeasible basis. Then we can show that $\xi_0 = \xi_{\max}$ in the same way as infeasible case.

Here we show the advantage of the criss-cross method for solving $P(\xi_0 + \Delta_{\min})$. When we apply a computer program for solving $P(\xi_0 + \Delta_{\min})$, we have to determine the input size of $\Delta_{\min}$. However, the above lemma indicates that the sign of each component of $\bar{d}_N + (\xi_0 + \Delta_{\min})\bar{c}_N$ and $q_N + (\xi_0 + \Delta_{\min})p_N$ can be checked by the lexicographical ordering between $(\bar{d}_j + \xi_0 \bar{c}_j, \bar{c}_j)$, $(q_j + \xi_0 p_j, p_j)$ and $\mathbf{0}$. The criss-cross method is a simple finite algorithm for linear programming developed by Zionts [18] and extended to the setting of oriented matroid by Terlaky [15], [16] and Wang [17], and it requires only the sign of each component of the tableaux. Thus the above lemma implies that we can apply the criss-cross method for solving $P(\xi_0 + \Delta_{\min})$ without deciding the magnitude of $\Delta_{\min}$. Another advantage of the criss-cross method is that the initial basis need not be either feasible or dual feasible. In case $\beta_1 = \beta_2$, the current basis (optimal to $P(\xi_0)$) is neither feasible nor dual feasible for $P(\xi_0 + \Delta_{\min})$. However we can take the current basis as an initial basis of the criss-cross method.

Now we discuss the finite pivoting rules for the nonconvex quadratic programming problems treated in Section 3. In [7], Klafszky and Terlaky modified the criss-cross method for a quadratic programming problem with a symmetric positive semi-definite matrix. Similar to the ordinary criss-cross method, the modified algorithm requires only the sign of each component of the tableaux. This means that we can construct an algorithm for the nonconvex quadratic programming problems which avoids the cycling in the same way as discussed above.

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